Classic Decision Theory

Faruk Gul's ECO511 Lectures summary by N. Antić

At the start of the course we reviewed deterministic choice and at the end we covered Dekel's betweenness. These are not contained in this summary.

Mixture Spaces

- The primitives are an arbitrary set P and an operation $h : [0,1] \times P \times P \to P$, we will write $h_a(p,q)$ instead of $h(a,p,q)$
- $\langle P, h\rangle$ is a **mixture space** if the following axioms hold:

M1 $h_1(p,q) = p$ (sure mix)

M2 $h_a(p,q) = h_{1-a}(q,p)$ (commutativity)

M3 $h_a(h_b(p,q), q) = h_{ab}(p,q)$ (one-sided distributivity)

- **M1**, **M2**, **M3** $\Rightarrow$ **M4**: $h_c(h_a(p,q), h_b(p,q)) = h_{ca+(1-c)b}(p,q)$ (two-sided distributivity)
- **M1**, **M2**, **M3** $\Rightarrow$ **M0**: $h_a(p,p) = p$ (trivial mix)
- Careful! The following properties are true in lottery spaces, but **not true** in general mixture spaces:
  - $h_a(h_b(p,q), r) = h_{ab}\left(p, h_{\frac{a(1-b)}{1-ab}}(q,r)\right)$ (associativity)
  - $h_a(p,r) = h_a(q,r) \Rightarrow p = q$ (determinicity)
- A binary relation $\succeq$ on P satisfies the von Neumann–Morgenstern (vNM) axioms if:

A1 $\succeq$ is a preference relation (complete and transitive)

A2 $p \succ q, a \in (0,1) \Rightarrow h_a(p,r) \succ h_a(q,r)$

A3 $p \succ q \succ r \Rightarrow \exists a, b \in (0,1) : h_a(p,r) \succ q \succ h_b(p,r)$

- Given **A1** and **A2**, the following is equivalent to **A3** if the set P is a topological space:

A3* For every $p \in P$, the sets $\{q : p \succeq q\}$, $\{q : q \succeq q\}$ are closed

Theorem (Mixture Space Theorem, Herstein-Milnor 1953)**.** *Let $\langle P, h\rangle$ be a mixture space and $\succeq$ binary relation on P. Then:*

$\succeq$ satisfies **A1**, **A2**, **A3** *iff $\succeq$ has a linear representation.*

The representation is unique up to an affine transformation, i.e., if $U : P \to \mathbb{R}$ is a linear function that represents $\succeq$ and V is some other function that represents $\succeq$ then $V = aU + b$, for $a > 0$, $b \in \mathbb{R}$.

- U is **linear** if $U(h_a(p,q)) = aU(p) + (1-a)U(q)$
- Before proving the theorem, we showed two lemmas

Lemma 4.1. *If $\succeq$ satisfies A1, A2, A3 then:*

1. $1 \geq a > b \geq 0$, $p \succ r \Rightarrow h_a(p,r) \succ h_b(p,r)$
2. $a \in (0,1)$, $p \sim q \Rightarrow h_a(p,q) \sim q$; *if* $p \succ q$ *then* $p \succ h_a(p,q) \succ q$
3. $p \succeq q, r \succeq s \Rightarrow h_a(p,r) \succeq h_b(q,s)$ *strict if strict.*

Lemma 4.2. *Let $\alpha(p,q,r) := \sup\{a \in [0,1] : q \succ h_a(p,r)\}$. For any p,q,r such that $p \succ q \succ r$, $h_{\alpha(p,q,r)}(p,r) \sim q$.*

- We proved the theorem by construction
- Note that if $p \sim q$ for all $p, q \in P$ then we can set $U(p) = 1$ for all p and we are done
- Thus assume $\overline{p} \succ \underline{p}$ for some $\overline{p}, \underline{p} \in P$ and for any $r \in P$ define:

$$U(r) = \begin{cases} \frac{1}{\alpha(r,\overline{p},\underline{p})} & \text{if } r \succ \overline{p} \\ 1 & \text{if } r \sim \overline{p} \\ \alpha(\overline{p}, r, \underline{p}) & \text{if } \overline{p} \succ r \succ \underline{p} \\ 0 & \text{if } r \sim \underline{p} \\ \frac{-\alpha(\overline{p},\underline{p},r)}{1-\alpha(\overline{p},\underline{p},r)} & \text{if } \underline{p} \succ r \end{cases}$$

  - Consider each case in turn to show that U represents $\succeq$
- To show that U is linear, we again consider the various cases
  - Assume $\overline{p} \succ p \succ \underline{p}$, $\overline{p} \succ q \succ \underline{p}$ (case 3 for both)
  - Note that $h_{U(p)}(\overline{p}, \underline{p}) \sim p$, and $h_{U(q)}(\overline{p}, \underline{p}) \sim q$, by 4.2 and construction of U
  - By 4.1(iii) and the above (as well as M4):

$$\begin{aligned} h_a(p,q) &\sim h_a\left(h_{U(p)}(\overline{p},\underline{p}), h_{U(q)}(\overline{p},\underline{p})\right) \\ &= h_{aU(p)+(1-a)U(q)}(\overline{p},\underline{p}), \end{aligned}$$

  - Thus $U(h_a(p,q)) = aU(p) + (1-a)U(q)$
- That $\succeq$ is cts and U represents $\succeq$ does *not* imply that U is cts
  - If in addition to the above U is also linear, then U must also be continous
- The representation is unique in the followin sense

Theorem. *If U is a linear function that represents $\succeq$ and $V \neq U$ is linear, then V represents $\succeq$ iff $\exists a > 0$, $b \in \mathbb{R}$ such that $V = aU + b$.*

von Neumann-Morgenstern

- Let X be a *finite* set of prizes, $\mathcal{L}(X)$ be lotteries over X
- An expected utility representation is a utility function U which represents $\succeq$ over $\mathcal{L}(X)$, such that there exists some $u\colon X \to \mathbb{R}$ such that $U(p) = \sum_{x\in X} u(x)p(x)$ for any $p \in \mathcal{L}(X)$

Theorem (von Neumann-Morgenstern 1947)**.** *$\succeq$ on $\mathcal{L}(X)$ satisfies* **A1**, **A2**, **A3** *iff it has an expected utility representation.*

- Proof is a consequence of the mixture space theorem and the following lemma

Lemma. *If U is a linear function on $\mathcal{L}(X)$, then there exists some $u\colon X \to \mathbb{R}$ such that $U(p) = \sum_x u(x)p(x)$ for all $p \in \mathcal{L}(X)$. Conversely, if $U(p) = \sum_x u(x)p(x)$ for all $p \in \mathcal{L}(X)$, then U is linear.*

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  - Proof of lemma is by induction on the number of prizes

von Neumann-Morgenstern on Monetary Prizes

- X *infinite* set of prizes, $X = [w, b] \subset \mathbb{R}$, $w < b$
- Let $\mathbf{F}$ be the set of CDFs on X
- The axioms are slightly adjusted, as follows:

A1+M $\succeq$ is a preference relation and satisfies monotonicity, i.e., $x > y$ implies $\delta_x \succ \delta_y$

A2 $p \succ q, a \in (0,1) \Rightarrow h_a(p,r) \succ h_a(q,r)$

A3* For every $F \in \mathbf{F}$, the sets $\{G : G \succeq F\}$ and $\{G : F \succeq G\}$ are closed under the topology induced by the metric $d(F,G) = \int |F - G| \, dx$

- A3* implies we are endowing $\mathbf{F}$ with the weak topology
  - A3* can be replaced by other notions of weak convergence
  - e.g., if $F^n \to F$ at every continuity point of F and $F^n \succeq G$ for all n, then $F \succeq G$

Theorem. *$\succeq$ on $\mathbf{F}$ satisfies* **A1+M**, **A2**, **A3*** *iff there is a continuous, strictly increasing $u : X \to \mathbb{R}$, such that:*

$$U(F) = \int u(x) \; \mathrm{d}F(x) \quad \text{represents} \succeq .$$

- F second order stochastically dominates G if $F \neq G$ and $\int_{-\infty}^{z} G(x) \; \mathrm{d}x \geq \int_{-\infty}^{z} F(x) \; \mathrm{d}x$ for all z
- $\succeq$ is **risk averse** when $F \succ G$, if F second order dominates G

Theorem (Notions of Risk Aversion)**.** *Let $\succeq$ on* $\mathbf{F}$ *satisfy* **A1+M**, **A2**, **A3*** *and u be the vNM utility index. Then:*

$$\succeq \text{ risk averse} \Leftrightarrow u \text{ strictly concave} \Leftrightarrow \delta_{\frac{1}{2}x+\frac{1}{2}y} \succ \frac{1}{2}\delta_x + \frac{1}{2}\delta_y.$$

Anscombe-Aumann

- Let $S = \{1, ..., n\}$ be a finite set of states, X be a set of prizes
- Let H be a set of acts $f : S \to \mathcal{L}(X)$
- We have the following AA axioms:

AA1-AA3 These are just **A1–A3** for $\succeq$ on H

AA4 $x \succ y$ for some $x, y \in X$ (non degenerate preference)

AA5 For every $f, g, \hat{f}, \hat{g}$, such that there are non-null $i, j \in S$ so that $f_k = g_k$ for all $k \neq i$, $\hat{f}_k = \hat{g}_k$ for all $k \neq j$ and $f_i = \hat{f}_j$, $g_i = \hat{g}_j$, we have $f \succeq g$ implies $\hat{f} \succeq \hat{g}$ (state separability)

- A state $i \in S$ is *null* if for all f, g such that $f_j \equiv g_j$, for $j \neq i$ we have $f \sim g$
  - Note that **AA5** implies state-separable preferences

Theorem (Anscombe-Aumann 1963)**.** *$\succeq$ satisfies* **AA1 − AA5** *on H iff there exists a non-constant* linear *U on $\mathcal{L}(X)$, and a probability μ on S such that:*

$$W(f) := \sum_{s \in S} U(f_s) \, \mu(s)$$

represents $\succeq$. U is unique up to a positive affine transformation.

- The key part of the proof is the lemma below

Lemma. *Function $W \colon H \to \mathbb{R}$ is linear iff $\exists \{U_s\}_{s \in S}$, such that $W(f) = \sum_{s \in S} U_s(f_s)$.*

- The rest of the proof proceeds as follows:
  - Using **AA5** show that all U_s from the lemma represent the same preferences up to positive affine transformation
  - By **AA4** there is one non-null state $i \in S$
    - Let the positive affine transformations taking utility from state j to state i be $a_j > 0$, b_j
    - Define $\mu(j) = a_j$ and normalize to sum to 1
- We extended this to arbitrary S, but restricting to H^0, the set of *simple* acts, i.e., acts which yield finite number of prizes
- In this case we needed a slightly stronger axiom **AA5**:

AA5* $f, g \in H^0$, and $p, q \in \mathcal{L}(X)$, and non-null events $E, \hat{E}$, such that: $f_s = g_s$ for $s \notin E$, $\hat{f}_s = \hat{g}_s$ for $s \notin \hat{E}$, and $f_s = \hat{f}_{\hat{s}} = p$, $g_s = \hat{g}_{\hat{s}} = q$ for all $s \in E$, $\hat{s} \in \hat{E}$ we have $f \succeq g$ implies $\hat{f} \succeq \hat{g}$

- Event $E \subset S$ is null if $f_s = g_s$, $\forall s \in S \setminus E$ implies $f \sim g$

Qualitative Probability

- We began by looking at some facts from probability theory

Fact 1. *A finitely-additive probability μ on A, an algebra on S, can be extended to 2^S.*

Fact 2. *A σ-additive probability μ on A, an algebra on S, can be extended to $\sigma(\mathcal{A})$.*

- A probability μ on an algebra $\mathcal{A}$ is **convex-valued** if $\forall A \in \mathcal{A}$, $r \in [0,1]$, there is $B \subset A$ such that $\mu(B) = r\mu(A)$
- A probability μ on an algebra $\mathcal{A}$ is **non-atomic** if $\forall \mu(A) > 0$, there is $B \subset A$ such that $0 < \mu(B) < \mu(A)$

Theorem. *Convex-valued $\mu \Rightarrow$ non-atomic μ. The converse is true for σ-additive μ.*

- Preference $\succeq^*$ over $\mathcal{A}$ is a qualitative probability if:

Q1 $\succeq^*$ is complete and transitive

Q2 $A \succeq^* \varnothing$, for all A

Q3 $S \succ^* \varnothing$

Q4 $A \succeq^* B$ iff $A \cup C \succeq^* B \cup C$, when $A \cap C = B \cap C = \emptyset$

- Further, there was an important axioms regarding partitions

P $A \succ^* B$ implies $\exists \{A_1, ..., A_n\}$, a partition of S, such that $A \succ^* (B \cup A_i)$ for every A_i

  - Says that not too many things are different from each other"

- Kreps book shows that a qualitative probability, $\succeq^*$, satisfies **P** iff it is both fine and tight:
  - $\succeq^*$ is **fine** if for all $A \succ^* \varnothing$, there is a finite partition of S no member of which is as likely as A
  - $\succeq^*$is **tight**, if for all $A \succ^* B$, there is C such that $A \succ^* (B \cup C) \succ^* B$
- Below are some examples of the above theorem
  - Let $S = [0,1] \cup [2,3] = S_1 \cup S_2$ and write $A = A_1 \cup A_2$ (where $A_1 \subseteq S_1, A_2 \subseteq S_2$)

Example (Lexicographic)**.** *$A \succ^* B$ iff $|A_1| > |B_1|$ or if $|A_1| = |B_1|$ and $|A_2| > |B_2|$*

- Fineness fails: For $A = [2, 2.5]$, we have $A \succ^* \varnothing$, by every finite partition includes an element with a positive mass on S_1

Example (Substitutes)**.** *$A \succ^* B$ iff $|A_1| + |A_2| > |B_1| + |B_2|$, or if $|A_1| + |A_2| = |B_1| + |B_2|$ and $|A_1| > |B_1|$*

- Tightness fails: For $A = [0, 0.5]$, $B = [2, 2.5]$, we have $A \succ^* B$, but nothing can be added to B that will not make it strictly more likely than A

Theorem. *Let $\succeq^*$ satisfy* **Q1–Q4** *and* **P***. Then $\exists \mu$ a finitely-additive prob. that represents $\succeq^*$; μ is unique and convex valued.*

Fact*. Every $\succeq^*$ satisfying **Q1–Q4** and **P** has an 2^n-equipartition (for every n), and for $A \succ^* B$, there is $C \subset A$ such that $C \sim^* B$.

- The proof of the theorem uses this fact and a few lemmas

Lemma 0. *Assume $A \cap B = \emptyset = C \cap D$. If $A \succeq^* C$ and $B \succeq^* D$, then $A \cup B \succeq^* C \cup D$. Further $A \succ^* C$ and $B \succ^* D$* implies $A \cup B \succ^* C \cup D$.

Lemma 1. *Let $a = \{A_1, \ldots, A_n\}, b = \{B_1, \ldots, B_m\}$ be two equipartitions of S. Then (i) $n = m$ implies $A_i \sim^* B_j$ for all i, j (ii) $n > m$ implies $A_i \succ^* B_j$ for all i, j (iii) $n = 2m$ implies $A_i \sim^* B_j \cup B_k$.*

- Define $k(n, a, A) = \min\left\{k : \cup_{i=1}^{k} A_i \succeq^* A\right\}$, for some $a = \{A_i\}_{i=1}^{2^n}$
- Further define $k(n, A) = \min_{a \in \mathbf{a}^{2^n}} k(n, a, A)$
  - By lemma 1, $k(n, A) = k(n, a, A)$
- Finally define $\mu(A) = \lim_n \frac{k(n,A)}{2^n}$

Lemma 2. *$\mu(A)$ is a finitely-additive probability.*

- The final step is to show μ represents $\succeq^*$and is unique and convex valued

Savage

- Let S be an artbitrary set of states, X the set of prizes
- An act $f \colon S \to X$ is **simple** if it takes a finite number of prizes
- Let $\succeq$ be a relation on F, the set of all simple acts
  - For any $x \in X$, let $x \in F$ be the act always returning prize x
  - Let fAg denote the act which is the same as f on states $A \subset S$ and the same as g on $S \setminus A$
- The Savage axioms are:

  S1 $\succeq$ is a preference relation

  S2 There is some $x, y \in X, F$ such that $x \succ y$ (non-degeneracy)

  S3 $fAh \succeq gAh$ implies $fAh' \succeq gAh'$ (sure-thing principle)

  S4 If A is non-null, then $xAh \succeq yAh$ $\forall h$ iff $x \succeq y$ (Kreps sure-thing principle)

  S5 $xAy \succeq xBy$ implies $x'Ay' \succeq x'By'$, for all $x \succ y$, $x' \succ y'$ (states are independent of prizes)

  S6 For all $f \succ g$, $x \in X$ there is a finite partition of S such that $xAf \succ g$, and $f \succ xAg$ (strong continuity)

- Some useful observations
  - For **S6** to be satisfied S must be infinite
  - For **S5** to be violated we need at least 3 prizes
  - The existence of an additive representation implies **S3**
  - The strict version of **S4** is for all non-null A: $x \succ y$ iff $\exists h. xAh \succ yAh$
  - The "for all h" is important for constructing counter-examples of **S4**
  - In the absence of completeness, zero measure sets might not be null, as by definition, A is null if for all f, g, h: $fAh \sim gAh$, but the latter may not be comparable

Theorem (Savage 1954)**.** *A binary relation $\succeq$ on the set of simple savage acts satisfies S1-S6 iff there exists a non-constant function u on X, and a finitely-additive probability on $(S, 2^S)$, such that $W(f) = \sum_{x \in X} u(x) \mu(f^{-1}(x))$ represents $\succeq$. Furthermore, μ is unique and u is unique up to an affine transformation.*

- Steps of the proof:
  - Pick any $x \succ y$ and define $\succeq^*$ by $A \succeq^* B$ iff $xAy \succeq^* xBy$.
  - $\succeq^*$ is a qualitative probability that satisfies Axiom P, hence there exists a uniqe convex valued finitely-additive probability μ that represents $\succeq^*$
  - For $f \in F$, define p_f by $p_f(x) = \mu(f^{-1}(x))$, i.e. fold-down (simple) acts to (simple) lotteries
  - Show that $\phi : F \to \mathcal{L}^0(X)$, defined by $\phi(f) = p_f$ is onto, i.e. every (simple) lottery $p \in \mathcal{L}^0(X)$ has an act that "reduces" to it
  - $p_f = p_g$ implies $f \sim g$
  - Define $\succeq^0$ on $\mathcal{L}^0(X)$ by $p \succeq^0 q$ iff there are f, g such that $f_p = p$, $g_p = q$, and $f \succeq g$
  - $\succeq^0$ satisfies (vNM) A1-A3 and therefore there exists a function $u \colon X \to \mathbb{R}$ that represents $\succeq^0$ on $\mathcal{L}^0(X)$
  - Combined together, u, μ yield the desired representation: $W(f) = \sum_{x \in X} u(x) \mu(f^{-1}(x))$

Comments about Finiteness Assumptions

- Throughout this course we worked with finite primitive sets, with occasional excursions into the infinite, in particular:
  - vNM: X is a finite set of prizes
  - vNM over monetary prizes: here we considered infinite X, but with the natural order structure, allowing us to approximate elements in $\mathcal{L}(X)$ by simple lotteries, using an added monotonicity requirement
  - A-A: S is finite and in the homework we extended the result to infinite S but restricted to *simple* acts, i.e. acts that give a finite number of outcomes (lotteries in $\mathcal{L}(X)$ in this case)
  - Savage: S is infinite (required for **P** to hold), but we restrict attention to *simple* savage acts